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COMMENT

On 'conflict of conservation laws in cyclotron radiation'

Patrick DasGupta
Tata Institute of Fundamental Research, Bombay-400 005, India

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Abstract. It is shown that conservation of energy, linear momentum and angular momentum are all compatible with each other in the case of an electron undergoing cyclotron emission in a uniform and constant magnetic field. We also point out the flaw in the argument of Lieu *et al* claiming the incompatibility of the conservation principles.

The problem of an electron emitting cyclotron radiation in the presence of a uniform and constant magnetic field has been considered. Recently it has been claimed (Lieu *et al* 1983) that in the above process conservation of angular momentum and energy is not compatible with the conservation of linear momentum. This claim has serious repercussions not only on the fundamental structure of physics, conservation laws being intimately related to symmetry principles, but also on the dynamics of the large-scale structure of the universe, because many of the astrophysical objects like supernova remnants and radio jets are believed to be radiating by means of cyclotron or synchrotron processes. Therefore, we chose to look into the problem more closely. From our investigations we conclude that the conservation laws are strictly maintained as far as cyclotron emission is concerned.

The system under consideration is an electron undergoing cyclotron emission in a uniform and constant magnetic field (z direction by convention).

A consequence of the conservation of energy and angular momentum (Lieu *et al* 1983, henceforth referred to as LLE) of the system is that

$$\mathrm{d}r_0^2/\mathrm{d}t = 0 \tag{1}$$

where $r_0^2 = x_0^2 + y_0^2$ is the radial position of the electron guiding centre. Similarly, one can show (equations (7)–(10) of LLE) that conservation of linear momentum implies

$$\mathrm{d}r_0^2/\mathrm{d}t = (mw_\mathrm{c})^{-2}\,\mathrm{d}P^2/\mathrm{d}t \tag{2}$$

where w_c is the cyclotron frequency, m is the mass of the electron, and $P^2 = P_x^2 + P_y^2$, P_x and P_y being the x and y components respectively of the radiation field.

Now comes the crucial point. LLE proceed to show that conservation of energy and angular momentum is not compatible with the conservation of linear momentum, by demonstrating that $\mathrm{d}P^2/\mathrm{d}t$ is non-zero, thus giving rise to a conflict between (1) and (2).

We do not agree with the above claim made by LLE because, as we will shortly demonstrate, $\mathrm{d}P^2/\mathrm{d}t$ is zero whenever there is an azimuthal symmetry. We give the following simple and straightforward argument.

It can be shown (as in the appendix of LLE) that for an electron undergoing cyclotron emission,

$$\mathrm{d}P_x/\mathrm{d}t = \mathrm{d}P_y/\mathrm{d}t = 0. \tag{3}$$

This means that P_x and P_y are constant in time. Hence P^2, which is defined to be the sum of P_x^2 and P_y^2, is necessarily constant in time. Another way of putting this is the following,

$$\mathrm{d}P^2/\mathrm{d}t = 2P_x\,\mathrm{d}P_x/\mathrm{d}t + 2P_y\,\mathrm{d}P_y/\mathrm{d}t, \tag{4}$$

which is identically zero by virtue of (3).

Clearly (4) implies that there is no conflict between (1) and (2), so that the *conservation laws are all compatible with each other.*

Next we show where LLE had gone wrong. We begin with the expression for the energy radiated in a given harmonic m, per unit solid angle and per unit time (Bekefi 1966),

$$j(w, \theta) = (e^2w^2/8\pi^2\varepsilon_0 c)[\cot^2\theta\, J_m^2(m\beta\sin\theta) + \beta^2 J_m'^2(m\beta\sin\theta)] \tag{5}$$

where $\beta = v/c$, v is the (transverse) velocity of the electron, θ is the pitch angle of radiation propagation and

$$w = mw_c.$$

Let $p_x(w, \theta)$ and $p_y(w, \theta)$ be the x and y components respectively of the momentum of radiation in the given harmonic m flowing through the solid angle $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ centred around the direction given by (θ, ϕ). It is then clear that

$$\mathrm{d}p_x(w, \theta)/\mathrm{d}t = c^{-1}j(w, \theta)\sin\theta\cos\phi\,\mathrm{d}\Omega, \tag{6a}$$

$$\mathrm{d}p_y(w, \theta)/\mathrm{d}t = c^{-1}j(w, \theta)\sin\theta\sin\phi\,\mathrm{d}\Omega. \tag{6b}$$

Since

$$P_\alpha = \sum_m \sum_{\substack{\text{over all}\\ \text{solid angles}}} p_\alpha(w, \theta) \tag{7a}$$

it is true that

$$\frac{\mathrm{d}P_\alpha}{\mathrm{d}t} = \sum_m \sum_{\substack{\text{over all}\\ \text{solid angles}}} \frac{\mathrm{d}p_\alpha}{\mathrm{d}t}(w, \theta) \tag{7b}$$

where $\alpha = x, y$. But by *no* means is

$$\mathrm{d}P/\mathrm{d}t = \sum\sum[(\mathrm{d}p_x/\mathrm{d}t)^2 + (\mathrm{d}p_y/\mathrm{d}t)^2]^{1/2} \tag{8a}$$

true (hereafter, $\Sigma\Sigma$ stands for $\Sigma_m\,\Sigma_{\text{overall solid angles}}$). And, it is (8$a$) which has been used by LLE to obtain the last expression in their appendix, i.e.

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{e^2}{4\pi\varepsilon_0 c^2}\sum_{m=1}^{\infty} w^2 \int_0^\pi \mathrm{d}\theta[\cos^2\theta\, J_m^2(m\beta\sin\theta) + \beta^2\sin^2\theta\, J_m'^2(m\beta\sin\theta)]. \tag{8b}$$

These faulty expressions (8a, b) are the cause for the apparent paradox. Now we will give the correct expression for $\mathrm{d}P/\mathrm{d}t$.

$$P \equiv (P_x^2 + P_y^2)^{1/2} = [(\Sigma\Sigma\, p_x(w, \theta))^2 + (\Sigma\Sigma\, p_y(w, \theta))^2]^{1/2}. \tag{9}$$

Differentiating (9) with respect to time yields

$$\frac{\mathrm{d}P}{\mathrm{d}t}=\frac{1}{P}\left[\left(\sum\sum p_x\right)\left(\sum\sum\frac{\mathrm{d}p_x}{\mathrm{d}t}\right)+\left(\sum\sum p_y\right)\left(\sum\sum\frac{\mathrm{d}p_y}{\mathrm{d}t}\right)\right]$$

$$=P^{-1}\left(P_x\,\mathrm{d}P_x/\mathrm{d}t+P_y\frac{\mathrm{d}P_y}{\mathrm{d}t}\right). \tag{10}$$

The last step in (10) follows from (7a) and (7b). Of course (3) and (10) both imply

$$\mathrm{d}P/\mathrm{d}t=0,$$

thereby resolving the paradox.

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References

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